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Aspects of supertubes

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Abstract: We find supersymmetric solutions of the D4-brane Born-Infeld action describing D2 supertubes ending on an arbitrary curve inside a D4-brane. From the D4-brane point of view, these are dyonic strings. We also consider various higher dimensional extensions of the usual supertubes, involving expanded D4- and D3-brane configurations. Finally, considering the worldsheet theory for open strings on a supertube, we show that this configuration is an exact solution to all orders in α'. Further the causal structure of the open-string metric provides new insight into the arbitrary cross-section of the supertube solutions. From this point of view, it is similar to the arbitrary profile that appears for plane waves.

Keywords: D-branes, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

Brane expansion is an interesting aspect of the physics of D-branes which has been found to occur in a variety of contexts. In most cases, this expansion is a dynamical effect that arises through the interaction of the D-branes with external supergravity fields [1]. Various examples of this effect include the polarization of D-branes in Wess-Zumino-Witten backgrounds [2], giant gravitons [3] and the supergravity resolution of singularities in $N = 1^*$ super-Yang-Mills theory [4]. A similar expansion was found to be possible by stabilising an ellipsoidal D0-D2 bound state with angular momentum [5]. This construction provided the first example where the expansion was not due to external fields. However, these configurations were not supersymmetric.

More recently, however, Mateos and Townsend [6] constructed supersymmetric configurations in which a D2-brane has expanded into a cylinder in flat, empty space. Again, in contrast to the original examples, these supertubes are supported against collapse solely by the excitation of internal fields in the D2-brane worldvolume theory. More precisely, the (static) electric and magnetic fields on the brane produce an angular momentum which stabilises the cylinder at a finite radius. Remarkably the cross-section of the cylinder can be an arbitrary curve embedded in eight-dimensional space transverse to the axis of the cylinder [7, 8, 9]. That is, given worldvolume coordinates t, x, ϕ, the configuration

$$\begin{aligned} &X_0 = t\,, \qquad X_9 = x\,, \qquad X_i = X_i(\phi)\,, \qquad i = 1, \ldots, 8 \\ &F = dt \wedge dx + B(\phi) dx \wedge d\phi\,, \end{aligned} \tag{1.1}$$

where $X_i(\phi)$ and $B(\phi) > 0$ are arbitrary functions, constitutes a solution of the full Born-

Infeld equations of the D2-brane worldvolume. Further, all of these solutions still preserve 1/4 of the type IIA supersymmetries. Here F denotes the field strength of the worldvolume gauge field and it is crucial that the electric field is one in string units ($F_{tx} = 1$). These electric and magnetic fields can be interpreted as indicating that fundamental strings and D0-branes (respectively) have been 'dissolved' into the D2 worldvolume. Further investigations revealed interesting T-dual configurations corresponding to helical D-strings [10]. Other aspects of the physics of supertubes can be found in [11, 12, 13]

Another related facet of D-brane physics is the possibility that a Dp-brane can, with the excitation of certain worldvolume fields, 'morph' into a Dp'-brane, of lower or higher dimensionality. For example, from Matrix theory [14] or the dielectric effect [1], a collection of D0-branes can expand in to various higher dimensional branes through non-commutative geometry. Similarly, the above supertube configurations (1.1) can be described as a non-commutative geometry within the worldvolume theory of the constituent D0-branes [8, 15].

Another example of D-brane 'morphing' is that D-strings ending on a D3-brane can be described with remarkable accuracy, from the point of the worldvolume theory of the D3-brane [16, 17], as a spike adorned with an appropriate magnetic field. Remarkably, a complementary description of this geometry as a 'fuzzy funnel' [18] is also provided by the worldvolume theory of the D-strings. Now we may consider supertubes, composed of D2-branes, fundamental strings and D0-branes. Either of the first two constituents can end on D4-branes in a supersymmetric way [19]. Similarly, the supersymmetry of D0- and D4-branes is compatible (see, e.g., [20]) and the D0's can dissolve in the D4 worldvolume [21]. This suggests that a supertube can end on a D4-brane while preserving 1/8 of the supersymmetry. We will show this result in fact holds with an explicit construction.

In the present paper, we examine various aspects of the physics of supertubes. First, in section 2, we illustrate that supertubes can end on D4-branes by constructing the appropriate solution of the D4-brane worldvolume action, following [16, 17]. We verify that these configurations solve the full nonlinear Born-Infeld equations and that they preserve the expected 1/8 of the supersymmetries. The next two sections describe attempts to produce nontrivial higher dimensional extensions of the supertube. T-duality easily allows one to construct supertubes which are expanded Dp-branes where the spatial geometry is a flat cylinder of the form $S^1 \times R^{p-1}$. In section 3, we show that in the D4-brane case this geometry can be deformed such that the worldvolume metric is no longer flat. In the following section, we construct new D3-brane solutions where the spatial geometry is $S^1 \times S^1 \times R^1$, where the two orthogonal circles are each supported by independent angular momenta. Unfortunately the latter configurations are not supersymmetric. In section 5, we examine the world sheet theory of open strings ending on a supertube and show that it is conformal to all orders in α'. We discuss how the peculiar structure of the open-string metric [22] provides new insight into the arbitrary shape and magnetic field profile of the supertube (1.1). From this point of view, this result is similar to the arbitrariness in choosing the profile of plane wave excitation of the transverse scalars propagating on a Dp-brane. There is also a close analogy to the arbitrary profile appearing in certain exact closed string backgrounds representing gravitational waves [23]. Finally we give a brief discussion of our results in section 6.

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2. D2 supertubes ending on D4-branes

As discussed above, one should expect that supertubes can end on an orthogonal D4-brane while preserving 1/8 of the supersymmetry. We verify this intuition with an explicit construction of such a configuration within the worldvolume theory of the D4-brane. At leading order, the low energy theory on a single D4-brane reduces to ordinary Maxwell theory coupled to a set of massless scalars describing the transverse position of the brane — see, e.g., [20]. We begin by constructing an appropriate solution of this leading order theory. In the next subsection, we show that this configuration in fact solves the full Born-Infeld equations of motion. Subsequently we will also verify that the solution also preserves the expected supersymmetries, also at the non-linear level (κ-symmetry).

So let us consider a curve $\mathcal{C}$ inside a flat D4-brane extending in the directions (X_1, X_2, X_3, X_4) and find the gauge field configuration that describes a supertube extending along X_9 and ending on $\mathcal{C}$. In the D4 worldvolume, we use coordinates x_a, $a = 0, 1, \ldots, 4$. Space-like indices are denoted as $i, j, \ldots = 1, \ldots, 4$.

As described in the introduction, the supertube has three types of constituent branes: D2- and D0-branes, and fundamental strings. Our construction will have to incorporate all of these components by exciting the appropriate electromagnetic fields, as well as a displacement of the worldvolume which is accomplished by exciting the transverse scalar $X_9(x_a)$. Following [16], the displacement and the fundamental strings can be described by the following supersymmetric configuration

$$F_{0i} = -\partial_i A_0 = \partial_i X_9(\vec{x})\,, \qquad \Delta X_9(\vec{x}) = -4\pi^2 \rho(\vec{x})\,, \tag{2.1}$$

where the charge density $\rho(\vec{x}) > 0$ has support on $\mathcal{C}$ and corresponds to the local density of strings ending on $\mathcal{C}$. The D2-branes will be represented by a (static) magnetic field which has the curve $\mathcal{C}$ as a monopole source:

$$F' = \hat{*} dA\,, \qquad d\hat{*} dA = -4\pi^2 \hat{*} j\,, \tag{2.2}$$

with j a conserved current tangential to, and with support on, the curve $\mathcal{C}$. The symbol $\hat{*}$ denotes the Hodge dual in the four-spatial dimensions using flat metric δ_{ij}. Since j is conserved, $|j|$ is a constant corresponding to the number of D2-branes ending on $\mathcal{C}$, as follows from the Chern-Simons coupling to the corresponding RR field strength. If we look at the linearized supersymmetry conditions for the magnetic field

$$\delta\chi = F_{ij}\Gamma^{ij} = 0\,, \tag{2.3}$$

we find out that this configuration is not supersymmetric but becomes so if we add new components such that the magnetic field is self-dual, i.e., $F = \hat{*}F$. Therefore, we add to F' a dual magnetic field:

$$F = dA + \hat{*} dA\,, \qquad d\hat{*} dA = -4\pi^2 \hat{*} j\,. \tag{2.4}$$

One easily verifies that the linearized equations of motion (i.e., $d{*}F = 0 = dF$) are satisfied everywhere away from $\mathcal{C}$. Notice that j can now also be considered as an electric current sourcing the dA component of the magnetic field. The self-duality of this magnetic field also implies that $F \wedge F \neq 0$ which corresponds to the desired appearance of a density of D0-branes.[1]

The solution can be written explicitly as:

$$\begin{aligned} -A_0 = X_9(\vec{x}) &= \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^2}\, d^4x' \,, \\ A_i(\vec{x}) &= \int \frac{j_i(\vec{x}')}{|\vec{x}-\vec{x}'|^2}\, d^4x' \,, \end{aligned} \tag{2.5}$$

which satisfies also the gauge condition $\partial_a A^a = 0$. Notice that the self-duality condition for the spatial part of F, and the fact that $A_0 = -X_9$, are required by supersymmetry but not by the linearized equations of motion. Below we will find that these conditions also play an important role in simplifying the full non-linear Born-Infeld equations.

2.1 Born-Infeld equations

The Born-Infeld lagrangian which controls the low energy dynamics of the D4-brane world-volume theory may be written as:

$$\mathcal{L} = -\tau_4 \sqrt{-|g+F|}\,, \tag{2.6}$$

where $\tau_4 = \frac{1}{(2\pi)^{3/2} g_s}$ is the D4-brane tension — we have introduced units where the fundamental string tension is unity, i.e., $2\pi\alpha' = 1$. The induced metric on the worldvolume is $g_{ab} = \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$ and $|\cdots|$ is used to denote the determinant of the enclosed matrix. The full nonlinear equations of motion of the worldvolume fields are then:

$$\begin{aligned} \partial_a \mathcal{M}^{[ab]} &= 0 \\ \partial_a \left(\mathcal{M}^{(ab)} \partial_b X^\mu \right) &= 0\,, \end{aligned} \tag{2.7}$$

where

$$\mathcal{M}^{ab} = \sqrt{|g+F|}\,\left((g+F)^{-1}\right)_{ab}\,. \tag{2.8}$$

For a generic static configuration, it is useful to write $g+F$ as:

$$g+F = \begin{pmatrix} -1 & E^t \\ -E & M \end{pmatrix}, \tag{2.9}$$

where we have introduced $E_i \equiv F_{0i}$. The inverse of $g+F$ is given by:

$$(g+F)^{-1} = \frac{1}{\Delta} \begin{pmatrix} -1 & E^t M^{-1} \\ -M^{-1}E & M^{-1}\Delta + M^{-1}E \otimes E^t M^{-1} \end{pmatrix}, \tag{2.10}$$

where $\Delta = 1 - E^t M^{-1} E$. For the present case, $M_{ij} = \delta_{ij} + E_i E_j + F_{ij}$ since $\partial_i X_9 = E_i$ by eq. (2.1). A useful fact for inverting M_{ij} is that a self dual field $F_{ij} = \frac{1}{2}\epsilon_{ijkl}F_{kl}$ satisfies $F_{ij}F_{jk} = -\frac{1}{4}F^2 \delta_{ik}$. Hence one finds

$$(\mathbf{1} + E\otimes E + \mathbf{F})^{-1} = \frac{1}{1+\frac{1}{4}F^2}\left\{ \mathbf{1} - \mathbf{F} - \frac{1}{1+E^2+\frac{1}{4}F^2}(E-\bar{E})\otimes(E+\bar{E}) \right\}. \tag{2.11}$$

[1] Certain related solutions describing bound states of D0-branes and fundamental strings stretching between two D4-branes were described in [24].

Here $\mathbf{F}$ is notation for F_{ij} as a 4×4 matrix and $F^2 \equiv F_{ij}F_{ij}$. Also we introduced for convenience $\bar{E}_i = F_{ij}E_j$, which satisfies $\bar{E}E = 0$, $\bar{E}\bar{E} = E^2F^2/4$.

Finally a simple computation reveals that the determinant of $g + F$ is independent of E and takes the value:

$$|g+F| = -\Delta\,|M| = -\frac{1+F^2/4}{1+E^2+F^2/4}\left(1+\frac{F^2}{4}\right)\left(1+E^2+\frac{F^2}{4}\right) = -\left(1+\frac{1}{4}F^2\right)^2 . \tag{2.12}$$

Putting everything together, the final result for $\mathcal{M}$ has remarkably simple form:

$$\begin{aligned}
\mathcal{M}^{00} &= -\left(1+E^2+\frac{1}{4}F^2\right)\\
\mathcal{M}^{0i} &= E_i + \bar{E}_i\\
\mathcal{M}^{i0} &= -E_i + \bar{E}_i\\
\mathcal{M}^{ij} &= \delta_{ij} - F_{ij}\,.
\end{aligned} \tag{2.13}$$

Note that supersymmetry conditions from the previous section, $E_i = \partial_i X_9$ and $F_{ij} = \frac{1}{2}\epsilon_{ijkl}F_{kl}$, were essential ingredients in producing this simple form. The equations $\partial_i \mathcal{M}^{[ij]} = 0$ are satisfied if $\partial_i E_i = 0$ and $\partial_i F_{ij} = 0$, which match the Maxwell equations appearing at lowest order. Hence we are assured that these equations are satisfied by the solution given in eq. (2.5). The equations $\partial_i\left(\mathcal{M}^{(ij)}\partial_j X^\mu\right) = 0$ give:

$$\begin{aligned}
&\mu = 0 \,:\, \partial_i\left(E_j F_{ji}\right) = \partial_i\partial_j X_9 F_{ji} + E_j \partial_i F_{ji} = 0\\
&\mu = j \,:\, \partial_i\left(\mathcal{M}^{(ij)}\right) = \partial_i \delta_{ij} = 0\\
&\mu = 9 \,:\, \partial_i\left(\mathcal{M}^{(ij)}\partial_j X_9\right) = \partial^2 X_9 = 0\,.
\end{aligned} \tag{2.14}$$

These are automatically satisfied except the last one, which corresponds to the leading order scalar equation, and so is satisfied by the given solution. Hence we conclude that our configuration (2.5) satisfies all of the full nonlinear equations coming from the Born-Infeld action.

2.2 Hamiltonian

It is also useful to compute the energy density. In order to do so we first compute the momentum conjugate to A_i, which is

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial E_i}\,. \tag{2.15}$$

Using the notation of eq. (2.9) we get that for such a generic static configuration

$$\Pi_i = \tau_4 \sqrt{\frac{\det M}{\Delta}}\left(M^{-1}\right)_{(ij)} E_j\,. \tag{2.16}$$

Using the properties of M that we described above, we find the simple result that, for our case, $\Pi_i = \tau_4 E_i$. Note that with this result, the momentum density circulating in the world volume is given simply by $T_{0i} = \Pi_j F_{ji} = -\tau_4 \bar{E}_i$. Near the supertube, we can consider a coordinate σ along the supertube and a radial coordinate in the transverse space. Then, from eq. (2.5) we find an electric field E_r and a magnetic field $F_{\sigma r}$ meaning that, in the vicinity of the supertube, there is a non-vanishing $\bar{E}_\sigma$. This gives a momentum density along the supertube as expected.[2]

The hamiltonian can also be computed with the result:

$$\mathcal{H} = \sqrt{\tau_4^2 \det M + \Pi^t \bar{M} \Pi}\,, \tag{2.17}$$

where $\bar{M} = (M^{-1}_{(ij)})^{-1}$, i.e. we take M invert it, symmetrize and then invert back. The result in general is different from M but can be computed with the same methods as before giving:

$$\bar{M} = \left(1 + \frac{1}{4}F^2\right)\mathbf{1} + E \otimes E - \frac{1}{1+E^2}\bar{E} \otimes \bar{E}\,. \tag{2.18}$$

The determinant of M follows from eqs. (2.10) and (2.12):

$$\det M = \left(1 + \frac{1}{4}F^2\right)\left(1 + E^2 + \frac{1}{4}F^2\right)\,. \tag{2.19}$$

Replacing in eq. (2.17) gives:

$$\mathcal{H} = \tau_4 + \tau_4 \frac{1}{4}F^2 + \frac{1}{\tau_4}\Pi^2\,. \tag{2.20}$$

Integrating $\mathcal{H}$ over the D4 worldvolume, the total energy has three separate (divergent) contributions. The first term above yields the energy due to the D4-brane tension. The term Π^2/τ_4 yields that from the fundamental strings and the $\tau_4 F^2/4$, the energy from the D0-branes. Note that the appearance of the latter two contributions, but not a separate contribution for the D2-branes, matches the results found in analysing the energy of the supertubes [6, 7]. The result that the total energy of the present configuration is a simple sum of a D4-brane contribution and a supertube contribution reflects the fact that 1/8 of the supersymmetries are still preserved. The energy integral diverges close to the supertube. In that region it is convenient to choose an affine parameter σ along the supertube and spherical coordinates r, θ, ϕ in the transverse space. Excluding a small region of radius $\epsilon(\sigma)$ around $\mathcal{C}$ and integrating by parts using the equations of motion we obtain that the leading contribution for $r \to 0$ region is:

$$H = \int r^2 dr\, d\Omega_2\, d\sigma\, \mathcal{H} \;=\; 4\pi^3 \tau_4 \int d\sigma \left(\frac{j^2}{\epsilon(\sigma)} + \frac{\rho}{\epsilon(\sigma)}\right) = 2\pi\tau_2 |j| \int d\sigma \left(\frac{|j|}{\rho} + \frac{\rho}{|j|}\right) X_9\,, \tag{2.21}$$

where we introduced $X_9 = \pi\rho/\epsilon$ in the vicinity of the singularity. Note that with our choice of units, the standard D2-brane tension is given by $\tau_2 = 2\pi\tau_4$. Hence, as expected, we find that the divergence is proportional to the distance, i.e., X_9, by which the spike extends above the D4 worldvolume. Identifying $2\pi|j|$ with the number of D2-branes as before, it follows that we may identify $|j|/\rho$ and $\rho/|j|$ with B and Π of the supertube analysis [6]. Then the expected relation $\Pi B = 1$ satisfied on the supertube follows. Note that we are using the normalization of [7], where an affine parametrization of the supertube is also used. We see that the freedom in choosing ρ is converted into the freedom in choosing B.

[2]We thank D. Mateos for related correspondence.

2.3 Example

The solution can be better understood by considering two examples.[3] One is the case where $\mathcal{C}$ is a straight line and the density of strings ρ is arbitrary, and other the case where $\mathcal{C}$ is a circle and ρ is constant (which is the original supertube of [6]).

For $\mathcal{C}$ a straight line we can take a coordinate x along $\mathcal{C}$, and spherical coordinates (r, θ, ϕ) in the transverse space. The integrals in eq. (2.5) give:

$$\begin{aligned} A_0 &= -X_9 = -\int_{-\infty}^{\infty} \frac{\rho(x'+x)}{x'^2+r^2} dx' = -\frac{1}{r}\int \frac{\rho(x+yr)}{1+y^2} dy \\ A_x &= \frac{\pi j}{r} \,. \end{aligned} \tag{2.22}$$

From here we can compute the field strength:

$$F = dx^0 \wedge dX_9 + \frac{\pi j}{r^2} dx \wedge dr + \pi j \sin\theta d\theta \wedge d\phi \,. \tag{2.23}$$

The component $F_{\theta\phi}$ (coming from $\hat{*}dA$) corresponds to a D2-brane boundary along x, and the fields $F_{0r} = \partial_r X_9$ and $F_{xr} = \pi j/r^2$ to the electric and magnetic field of the supertube, respectively.

The induced metric turns out to be:

$$ds^2 = -dx_0^2 + dx^2 + dr^2 + r^2 d\Omega_2^2 + dX_9^2 \,, \tag{2.24}$$

where $d\Omega_2^2$ is the line element on the two-sphere parametrised by θ, ϕ. It is clear that for $r \to \infty$ we have just the D4-brane. For $r \to 0$ we obtain from (2.22) that $X_9 \simeq \pi\rho(x)/r$ and then $dX_9 \simeq -(\pi\rho(x)/r^2)dr$. In this region it is useful to use $X_9 \simeq \pi\rho/r$ as a coordinate instead of r, and rewrite the field strength and metric as:

$$\begin{aligned} F &\simeq dx_0 \wedge dX_9 + \frac{j}{\rho} dx \wedge dX_9 + \pi j \sin\theta d\theta \wedge d\phi \\ ds^2 &\simeq -dx_0^2 + dx^2 + dX_9^2 \,. \end{aligned} \tag{2.25}$$

We see that the configuration reduces to a supertube with unit electric field and magnetic field equal to $j/\rho(x)$, as we inferred above from the analysis of the energy. The metric in the θ, ϕ directions is singular but the $F_{\theta\phi}$ component of the field strength makes the action $\sqrt{g+F}$ non-singular and equal to that of the supertube.

The integrals can be done easily for a circular supertube with uniform magnetic field which is the original supertube of [6]. Let us consider then a circle of radius R and introduce polar coordinates (r, ϕ) in the plane of the circle and (ρ, θ) for the two remaining D4-brane coordinates. The density $\rho(\vec{x})$ will be taken to be uniform, $\rho(\vec{x}) = \rho_0 \delta(r-R)\delta(\rho)/(2\pi\rho)$, with ρ_0 a constant. The result is:

$$\begin{aligned} X_9 &= -A_0 = \frac{2\pi R\rho_0}{\xi} \\ A &= \frac{\pi j}{R}\left(\frac{r^2+\rho^2+R^2}{\xi} - 1\right) d\phi \,, \end{aligned} \tag{2.26}$$

[3] We have been informed that these examples were studied independently by David Mateos and Selena Ng.

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where we defined $\xi^2 = (R^2+\rho^2+r^2)^2 - 4r^2R^2$. Near the circle ξ goes to 0 and the solution becomes

$$X_9 = -A_0 \simeq \frac{2\pi R\rho_0}{\xi}\,, \qquad A \simeq \frac{2\pi jR}{\xi}\,d\phi\,, \tag{2.27}$$

which is the same as for the flat supertube we described before, after proper identification of the coordinates, i.e., $\xi \simeq 2R\sqrt{\rho^2+(r-R)^2}$.

2.4 Supersymmetry

To complete the identification of the worldvolume solution with the configuration of a D2 supertube ending on a D4-brane we should check that they preserve the same supersymmetries. For the D4 worldvolume theory, the supersymmetry condition can be written as [25]:

$$\Gamma\epsilon = \epsilon\,. \tag{2.28}$$

In our case,

$$\Gamma = \frac{1}{\sqrt{|g+F|}}\left\{1+\frac{1}{2}F^{ab}\gamma_{ab}+\frac{1}{8}F^{ab}F^{cd}\gamma_{abcd}\right\}\Gamma_{11}\gamma_{01234}\,, \tag{2.29}$$

where we introduced the worldvolume γ-matrices defined in terms of the space-time matrices Γ_μ as:

$$\begin{aligned}\gamma_0 &= \Gamma_0\\ \gamma_i &= \Gamma_i + E_i\Gamma_9\,,\end{aligned} \tag{2.30}$$

Space-like indices should be risen and lowered with the metric:

$$g_{ij} = \delta_{ij} + E_iE_j\,. \tag{2.31}$$

Using the self-duality of F_{ij} a straightforward calculation gives:

$$\begin{aligned}F^{ab}\gamma_{ab} &= -\frac{2}{1+E^2}\Gamma_0\left(\not{E}+E^2\Gamma_9\right)+\frac{2}{1+E^2}E_i\bar{E}_j\Gamma_{ij}\\ F^{ab}F^{cd}\gamma_{abcd} &= 8\Gamma_0\bar{\not{E}}\,\Gamma_{1234}+\frac{2F^2}{1+E^2}\left(1+\Gamma_9\not{E}\right)\Gamma_{1234}\\ \gamma_{01234} &= \left(1+\Gamma_9\not{E}\right)\Gamma_{01234}\,,\end{aligned} \tag{2.32}$$

where we used the notation $\not{E} = E_i\Gamma_i$ and introduced again $\bar{E}_i = F_{ij}E_j$. This gives finally

$$\begin{aligned}\Gamma = {}&\Gamma_{11}\Gamma_{01234}+\\ &+\frac{1}{1+\frac{1}{4}F^2}\Gamma_0\Bigg[\left(\frac{1}{2}F_{ij}\Gamma_{ij}-\not{E}\,\Gamma_9+\bar{\not{E}}\,\Gamma_9+\bar{\not{E}}\,\not{E}\right)(\Gamma_{1234}-1)+\\ &\qquad+\left(\not{E}\,\Gamma_0\Gamma_{1234}+\bar{\not{E}}\,\Gamma_9+\bar{\not{E}}\,\not{E}\right)(1+\Gamma_0\Gamma_9\Gamma_{11})\Bigg].\end{aligned} \tag{2.33}$$

Since F_{ij} and E_i are not constant the only solutions of $\Gamma\epsilon=\epsilon$ are those satisfying the three projections:

$$\begin{aligned}\Gamma_{11}\Gamma_{01234}\epsilon &= \epsilon\\ \Gamma_0\Gamma_{11}\epsilon &= \epsilon \qquad (\text{or } \Gamma_{1234}\epsilon = \epsilon)\\ \Gamma_{09}\Gamma_{11}\epsilon &= -\epsilon\,.\end{aligned} \tag{2.34}$$

Hence we find that 1/8 of the supersymmetries are preserved and they match precisely with those expected for the D4-brane and the supertube. In particular, the latter two conditions match those of D0-branes and fundamental strings stretched along the X_9 axis. As expected, there is no separate projection which we might associate with the constituent D2-branes of the supertube [6].

3. D4 supertubes

In this and the next section, we consider higher dimensional configurations which can be thought of as nontrivial extensions of the supertube. A natural way to increase the dimension of the D-branes is to apply T-duality in directions transverse to the original supertube configuration (1.1). Suppose that such supertube extends along X_9 and the cross section $\mathcal{C}$ is embedded in the directions (X_1, X_2, X_3, X_4). Performing two T-dualities along X_5 and X_6, we obtain a D4 supertube with the supersymmetries of the fundamental strings along X_9 and that of D2-branes filling the X_5-X_6 plane. For the D2 supertube we can choose the magnetic field and the shape, which amounts to choosing a distribution of D0-branes. However the moduli space of the D2-branes in our D4 supertube is larger than that of the D0-branes. Not only can we choose their positions but also their orientations as a function of ϕ. For the resulting configuration to be supersymmetric the D2-branes must have a common supersymmetry. That will be the case if they are related by an SU(2) rotation [26]. We will consider a specific example in some detail to understand the procedure better. However, as is shown below, this case is singular at infinity since there the energy density diverges but we can consider similar examples where the D2-branes are wrapped on a compact cycle of some internal manifold which makes this problem disappear.

Consider then the following embedding:

$$\begin{aligned} X_1 + iX_2 &= Re^{i\phi}\,, & X_5 + iX_6 &= y_1 e^{i\phi}\,, & X_7 + iX_8 &= y_2 e^{i\phi} \\ X_0 &= t\,, & X_9 &= x\,, & X_{3,4} &= 0\,, \end{aligned} \tag{3.1}$$

and the worldvolume gauge field:

$$F = dt \wedge dx + B(\phi) dx \wedge d\phi\,, \tag{3.2}$$

with $B(\phi) > 0$. The induced metric is

$$ds^2 = -dt^2 + dx^2 + \left(R^2 + y_1^2 + y_2^2\right) d\phi^2 + dy_1^2 + dy_2^2\,. \tag{3.3}$$

The only difference with the T-dual of the D2 supertube is that $g_{\phi\phi}$ depends on the extra coordinates $y_{1,2}$. It interesting to observe that, as opposed to the D2 supertube, the induced metric is *not* flat. Nevertheless, it is easy to see that the BI equations are still satisfied. Computing again $\mathcal{M}^{ij} = \sqrt{|g+F|}((g+F)^{-1})_{ij}$ we obtain that the only non-vanishing components are:

$$\begin{aligned} \mathcal{M}^{00} &= -\frac{f + B^2}{B}\,, & \mathcal{M}^{0x} &= -\mathcal{M}^{x0} = \frac{f}{B}\,, & \mathcal{M}^{0\phi} &= \mathcal{M}^{\phi 0} = -1 \\ \mathcal{M}^{xx} &= \frac{f}{B}\,, & \mathcal{M}^{x\phi} &= -\mathcal{M}^{\phi x} = -1\,, & & \\ \mathcal{M}^{y_1 y_1} &= \mathcal{M}^{y_2 y_2} = B(\phi)\,, & & & & \end{aligned} \tag{3.4}$$

with $f = R^2+y_1^2+y_2^2$. The equations $\partial_i \mathcal{M}^{[ij]} = 0$ are satisfied since f and B are independent of x and t. The equations $\partial_i(\mathcal{M}^{(ij)}\partial_j X^\mu) = 0$ are also satisfied because B is independent of $y_{1,2}$. For example, the equations for $\mu = 5, 6$ reduce to:

$$\partial_\phi \left(\mathcal{M}^{\phi\phi}\partial_\phi X^{5,6}\right) + \partial_{y_1}\left(\mathcal{M}^{y_1 y_1}\partial_{y_1} X^{5,6}\right) = 0\,. \tag{3.5}$$

The second term vanishes if B and $\partial_{y_1} X^{5,6}$ are independent of y_1. This is satisfied if $X^{5,6}$ is linear in y_1, which (together with a similar condition for $X^{7,8}$) implies that the D2-branes are flat. The first term is zero since $\mathcal{M}^{\phi\phi} = 0$, which is due to the fact that $F_{tx} = 1$. This is crucial because it allows $X^{5,6}$ to depend on ϕ.

We see that the D2-branes can in fact point in arbitrary directions. However we need that all the D2-branes preserve some common supersymmetry, which is true if they are related by an SU(2) rotation [26]. The computation of Γ is easier than in the previous section and gives:

$$\Gamma = \frac{1}{B}\left(1 + \left(\gamma^{0x}\Gamma_{11} + B\gamma^{x\phi}\Gamma_{11}\right)\right)\Gamma_{11}\gamma_{tx\phi y_1 y_2}\,. \tag{3.6}$$

The condition $\Gamma\epsilon = \epsilon$ is satisfied if:

$$\begin{aligned} \Gamma_0\Gamma_9\Gamma_{11}\epsilon &= \epsilon \\ \Gamma_0\gamma_{y_1}\gamma_{y_2}\epsilon &= \epsilon\,. \end{aligned} \tag{3.7}$$

The latter condition reduces to

$$\begin{aligned} \Gamma_0\Gamma_6\Gamma_8\epsilon &= \epsilon \\ \Gamma_0\Gamma_5\Gamma_7\epsilon &= \epsilon\,. \end{aligned} \tag{3.8}$$

These conditions are equivalent to those for fundamental strings along X_1 and D2-branes along $X_{4,6}$ and $X_{5,7}$. Hence this configuration (3.1), (3.2) preserves 1/8 of IIA supersymmetry.

If we compute the hamiltonian we encounter a problem. Indeed the energy is given by:

$$H = \tau_4 \int d\phi\, dx\, dy_1\, dy_2 \sqrt{R^2+\vec{y}^2}\left(\frac{B(\phi)}{\sqrt{R^2+\vec{y}^2}} + \frac{\sqrt{R^2+\vec{y}^2}}{B(\phi)}\right). \tag{3.9}$$

We see from above that the energy density diverges as $|\vec{y}| \to \infty$. This is due to the fact that as we get away from the centre of the helix, the density of D2-branes decreases and then the D4-brane becomes critical (det $\sqrt{g+F} \to 0$). In the T-dual picture this corresponds to a brane that at infinity moves at the speed of light. Clearly this problem is associated with the infinite extent of the D2-branes and so it may be avoided by considering a compact configuration. If we compactify the some of the directions on a torus, K3 or Calabi-Yau manifold then part of the supersymmetry is preserved and we can wrap the D2-brane along some supersymmetric cycle. Again, the moduli of such cycle can vary as a function of ϕ as long as some common supersymmetry is preserved. If the two-cycle contains non-trivial S^1 cycles then the moduli space includes also Wilson lines. For example in the case of T^4 we can wrap the D2-brane in a genus g surface whose moduli space is $T^4 \times \mathrm{Symm}^g(T^4)$ [27]. This configuration can also be described as that of g intersecting supertubes preserving 1/8 of the supersymmetry. If we consider $K3$ then the moduli space for a genus g surface is given by $\mathrm{Symm}^{\mathrm{g}}(\mathrm{K3})$ [27, 28]. Again the moduli can vary as a function of ϕ giving a large number of supertubes constructions. On the other hand, these moduli spaces can be considered as being related to the position of a D0-brane in a T-dual picture [28]. It would be interesting to see if in the case of six-dimensional manifolds there are examples related to rotations as in flat space.

4. D3-tubes

Applying a single T-duality transverse to the original supertube, we get a D3-brane with D1-branes and fundamental strings dissolved at right angles. Embedded in flat space, the T-dual configuration will have spatial topology $S^1 \times R^2$, where the S^1 is supported by angular momentum. In the following, we consider a configuration with topology $S^1 \times S^1 \times R$ where the orthogonal circles are both supported by separate angular momenta. However, we will find that while this solution is stable, it is not supersymmetric.

Consider Minkowski space in the following coordinates

$$ds^2 = -dT^2 + dX^2 + dR_1^2 + R_1^2 d\Phi_1^2 + dR_2^2 + R_2^2 d\Phi_2^2 + dE_5^2\,, \tag{4.1}$$

where the R_i and Φ_i are radial and angular coordinates on two mutually orthogonal planes and E_5 denotes five dimensional euclidean space. Our D3-tube, so-called, will be a D3-brane with one extended (x) and two compact (ϕ_1, ϕ_2) spatial worldvolume directions, along with time (t). We embed it in Minkowski space, using static gauge to align the worldvolume and background coordinates as follows: $t = T$, $x = X$, $\phi_i = \Phi_i$ and we will allow the radii to vary in each plane $R_i = R_i(\phi_i)$. The tube sits, point-like, at the origin of the transverse E_5. The Born-Infeld action for this D3-brane will take the form

$$S = -\int \mathrm{d}^3\sigma \sqrt{-\det|g+F|} \tag{4.2}$$

(where we set $\tau_3 = 1$ in the following for convenience). With the embedding described above, the induced metric g_{ab} on the the worldvolume becomes:

$$g_{tt} = -1\,, \qquad g_{xx} = 1\,, \qquad g_{\phi_1\phi_1} = R_1^2 + {R_1'}^2\,, \qquad g_{\phi_2\phi_2} = R_2^2 + {R_2'}^2\,, \tag{4.3}$$

where the primes denote differentiation with respect to the appropriate angular coordinate, i.e., $R_i' = \partial_{\phi_i} R_i$.

To induce separate angular momenta on each circle, we consider the worldvolume gauge field

$$F = E\mathrm{d}t \wedge \mathrm{d}x + B_1 \mathrm{d}x \wedge \mathrm{d}\phi_1 + B_2 \mathrm{d}x \wedge \mathrm{d}\phi_2\,. \tag{4.4}$$

This corresponds to switching on an axial electric field, E, and introducing a magnetic flux, proportional to B_i, across each of the compact circles. We will assume that E is constant across the entire worldvolume and that each of the magnetic components only varies around its associated loop, i.e., $B_i = B_i(\phi_i)$. From a microscopic perspective, we may think of E as arising from a uniform density of dissolved fundamental strings running parallel to the axis of the tube. The magnetic fields are associated with dissolved D1-branes wrapping each of the orthogonal circles. As the system contains orthogonal D1-branes, one should expect that no supersymmetries are preserved. In fact, this intuition can be verified with a detailed calculation.

Having established our ansatz, we now write out the lagrangian density,

$$\mathcal{L} = -\sqrt{(1-E^2)\left(R_1^2 + {R_1'}^2\right)\left(R_2^2 + {R_2'}^2\right) + B_1^2\left(R_2^2 + {R_2'}^2\right) + B_2^2\left(R_1^2 + {R_1'}^2\right)}\,, \qquad (4.5)$$

and perform the standard analysis of the Euler-Lagrange equations, the components of the gauge potential A, and the transverse scalars corresponding to the radii R_i. Due to our assumptions about the uniformity of the various fields, the equations of motion for A_t, A_{ϕ_1} and A_{ϕ_2} are automatically satisfied. The equation of motion for the remaining component A_x becomes:

$$0 = \partial_{\phi_1}\left\{\mathcal{L}^{-1}B_1\left(R_2^2 + {R_2'}^2\right)\right\} + \partial_{\phi_2}\left\{\mathcal{L}^{-1}B_2\left(R_1^2 + {R_1'}^2\right)\right\}. \qquad (4.6)$$

For the scalar R_1, one finds:

$$\mathcal{L}^{-1}\left(\left(1-E^2\right)R_1\left(R_2^2 + {R_2'}^2\right) + R_1B_2^2\right) = \partial_{\phi_1}\left\{\mathcal{L}^{-1}\left(\left(1-E^2\right)R_1'\left(R_2^2 + {R_2'}^2\right) + R_1'B_2^2\right)\right\}. \qquad (4.7)$$

From the obvious symmetry of the lagrangian density (4.5), the equation of motion for R_2 follows from that for R_1 by the interchange $(1 \leftrightarrow 2)$ in all subscripts. Carrying out the derivatives and simplifying each equation in turn leads to

$$\begin{aligned}
&\{(1-E^2)\left(R_2^2 + {R_2'}^2\right) + B_2^2\}\left(R_2^2 + {R_2'}^2\right)\left(B_1'\left(R_1^2 + {R_1'}^2\right) - B_1R_1'\left(R_1 + R_1''\right)\right) + \\
&\qquad \left\{(1-E^2)\left(R_1^2 + {R_1'}^2\right) + B_1^2\right\}\left(R_1^2 + {R_1'}^2\right)\left(B_2'\left(R_2^2 + {R_2'}^2\right) - B_2R_2'\left(R_2 + R_2''\right)\right) = 0\,, \\
&\qquad \left\{(1-E^2)\left(R_2^2 + {R_2'}^2\right) + B_2^2\right\}\left(\mathcal{L}^{-1}\left(R_1 - R_1''\right) - R_1'\partial_{\phi_1}\mathcal{L}^{-1}\right) = 0\,, \\
&\qquad \left\{(1-E^2)\left(R_1^2 + {R_1'}^2\right) + B_1^2\right\}\left(\mathcal{L}^{-1}\left(R_2 - R_2''\right) - R_2'\partial_{\phi_2}\mathcal{L}^{-1}\right) = 0\,.
\end{aligned} \qquad (4.8)$$

It is apparent that each equation is satisfied if we choose

$$\left(1-E^2\right)\left(R_i^2 + {R_i'}^2\right) + B_i^2 = 0 \qquad (i = 1,2)\,. \qquad (4.9)$$

However, this is presumably not the only solution. We could choose to set the other bracket in each R_i equation to zero. Then the A_x equation reduces to

$$\frac{B_1'}{R_1^2 + {R_1'}^2} + \frac{B_1\left(R_1 - R_1''\right)}{R_1'\left(R_1^2 + {R_1'}^2\right)} = \frac{B_2'}{R_2^2 + {R_2'}^2} + \frac{B_2\left(R_2 - R_2''\right)}{R_2'\left(R_2^2 + {R_2'}^2\right)} = C\,, \qquad (4.10)$$

where C is a constant, because each of the other terms is a function of independent variables. Hence, solving the differential equation

$$\frac{B'}{R^2 + R'^2} + \frac{B\left(R - R''\right)}{R'\left(R^2 + R'^2\right)} - C = 0 \qquad (4.11)$$

will also provide a solution to all of the equations. However, as we were unable to make further progress towards solving this equation, we focus on the solutions given by eq. (4.9).

This solution states that $B_i^2 = (E^2-1)(R_i^2 + {R'_i}^2)$, which requires $E^2 \geq 1$ and further implies that

$$\frac{B_1^2}{R_1^2 + {R'_1}^2} = \frac{B_2^2}{R_2^2 + {R'_2}^2} = E^2 - 1\,. \tag{4.12}$$

We also have periodic boundary conditions on the B_i, the R_i, and their derivatives. Solutions are easily constructed by finding a cross-section $(R_1(\phi_1),\, R_2(\phi_2))$ that satisfies the periodic boundary conditions and that is associated, via eq. (4.9), with magnetic fields that also satisfy the boundary conditions (this translates to a condition on each $R''_i(\phi_i)$). Then the magnetic fields are determined up to a common factor E^2-1, which is arbitrary up to being non-negative.

Given that these configurations are not supersymmetric, an interesting question is to determine whether or not they are stable. To analyse this point, we change to the hamiltonian formalism. The only nontrivial canonical momentum in the problem is that associated with A_x:

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_x)} = \frac{\partial \mathcal{L}}{\partial E} = -E\mathcal{L}^{-1}\left(R_1^2 + {R'_1}^2\right)\left(R_2^2 + {R'_2}^2\right). \tag{4.13}$$

We then write the electric field in terms of its canonical momentum, or 'electric displacement', as

$$E = \Pi \sqrt{\frac{f_1 f_2 + B_1^2 f_2 + B_2^2 f_1}{f_1 f_2\left[\Pi^2 + f_1 f_2\right]}}\,, \tag{4.14}$$

where, for the sake of brevity, we have defined $f_i \equiv R_i^2 + {R'_i}^2$ for $i = 1, 2$. Subsequently, the hamiltonian density is written as

$$\mathcal{H} = \Pi E - \mathcal{L} = \sqrt{\frac{\left[\Pi^2 + f_1 f_2\right]\left[f_1 f_2 + B_1^2 f_2 + B_2^2 f_1\right]}{f_1 f_2}}\,. \tag{4.15}$$

Hamilton's equations then tell us, among other things, that the 'on-shell' expression for Π is

$$\Pi = E\sqrt{\frac{f_1 f_2}{E^2 - 1}}\,, \tag{4.16}$$

which can be verified by substituting eq. (4.9) into eq. (4.13). Since all the fields are independent of x, we can integrate the hamiltonian density over a cross-section to get an energy per unit length that is uniform along the tube:

$$H = \int \mathrm{d}\phi_1 \mathrm{d}\phi_2\, \sqrt{f_1 f_2}\mathcal{H} = \int \mathrm{d}\phi_1 \mathrm{d}\phi_2\, \sqrt{\left[\Pi^2 + f_1 f_2\right]\left[f_1 f_2 + B_1^2 f_2 + B_2^2 f_1\right]}\,. \tag{4.17}$$

Corresponding to each solution is a triple of conserved quantities — the fluxes of B_1, B_2 and Π across the torus:

$$\begin{aligned} N_i^D &= \int \mathrm{d}\phi_i \sqrt{f_i} B_i\,, \qquad (i = 1, 2) \\ N^F &= \int \mathrm{d}\phi_1 \mathrm{d}\phi_2 \sqrt{f_1 f_2}\,\Pi\,. \end{aligned} \tag{4.18}$$

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That is, for a given configuration, the number (density) of D1-branes wrapping each of the circles is fixed, as is the number (density) of fundamental strings along the x axis.

To test the stability of the solutions we should consider the functional second derivatives of H with respect to variations in all the fields. However, physical fluctuations will be constrained by the conservation of N_i^D and N^F. From $\delta N^F = \delta N_i^D = 0$ it follows that

$$\begin{aligned}
\delta\Pi &= \left(\frac{d}{d\phi_1}\frac{\Pi R_1' f_2}{\sqrt{f_1 f_2}} - \frac{\Pi R_1 f_2}{\sqrt{f_1 f_2}}\right)\frac{\delta R_1}{\sqrt{f_1 f_2}} + \left(\frac{d}{d\phi_2}\frac{\Pi R_2' f_1}{\sqrt{f_1 f_2}} - \frac{\Pi R_2 f_1}{\sqrt{f_1 f_2}}\right)\frac{\delta R_2}{\sqrt{f_1 f_2}}\,, \\
\delta B_i &= \left(\frac{d}{d\phi_i}\frac{B_i R_i'}{\sqrt{f_i}} - \frac{B_i R_i}{\sqrt{f_i}}\right)\frac{\delta R_i}{\sqrt{f_i}}\,.
\end{aligned} \tag{4.19}$$

Using these expressions, we regard Π and B_i as functionals of the radii when calculating the variations in H. For now, we consider two cases only: general (flux-preserving) fluctuations of a D3-tube with uniform radii, and uniform fluctuations of a D3-tube with a general cross-section. In the first instance we have $R_i' = 0$, and in the second we have $\delta R_i' = 0$; these two assumptions greatly simplify the functional differentiation. Evaluated on the solution space of eq. (4.9), the matrix of second derivatives for each case can be written as

$$\mathbf{H}_{ij} \equiv \frac{\delta^2 H}{\delta R_i \delta R_j} = \frac{8 R_1 R_2 \sqrt{E^2-1}}{2E^2-1}\begin{pmatrix} 2E^2 (R_1 f_2)/(R_2 f_1) & 1 \\ 1 & 2E^2 (R_2 f_1)/(R_1 f_2)\end{pmatrix}, \tag{4.20}$$

which has an obvious simplification in the case of uniform radii. Given that $E^2 \geq 1$, the eigenvalues of $\mathbf{H}$ can be shown to be non-negative, implying stability of the system against small (flux-preserving) fluctuations of the fields, despite the loss of supersymmetry.

5. Open strings on supertubes

As mentioned in the introduction, one can choose an arbitrary function $B(\phi)$ and an arbitrary shape (as a function of ϕ) for the supertube and still have a supersymmetric solution of the Born-Infeld equations. To gain further insight into this curious aspect of supertubes, we investigate the worldsheet theory of open strings ending on a supertube. From this point of view, this arbitrariness means that, for example, for any boundary term associated with a magnetic field $F_{X_1\phi}(\phi)$,

$$S_{\mathrm{bdy.}} = \int_{\partial\Sigma} d\tau \mathcal{V}_{X_1\phi}(\tau) \tag{5.1}$$

$$\mathcal{V}_{X_1\phi}(\tau) = \int dk \tilde{A}_1(k) e^{ik\phi}\partial_\tau X_1 \tag{5.2}$$

$\mathcal{V}_{X_1\phi}(\tau)$ must have conformal dimension one since in (1.1), any field $B(\phi)$ satisfies the equations of motion. The only way this can be is if ϕ is a 'null' field by which we mean the correlator $\langle\phi\phi\rangle$ vanishes. This would ensure that there is no anomalous dimension associated with $e^{ik\phi}$, and so it has conformal dimension zero for any k. This is analogous to the usual statement that $e^{ik_\mu X^\mu}$ has conformal dimension 0 if $k^2 = 0$, i.e., $\vec{k}$ is null. Certainly we expect that ϕ should be a standard worldsheet field with a nonvanishing propagator $\langle\phi\phi\rangle \propto g^{\phi\phi}$. However, while this intuition is appropriate in the bulk of the worldsheet, the above discussion refers to an interaction introduced on the boundary and so the relevant metric for the boundary correlator is the open-string metric as defined in [22]. The latter is modified by the background electric and magnetic fields and we will see below that it indeed produces the desired result $\langle \phi\phi \rangle \propto G^{\phi\phi} = 0$.

With this motivation in mind, we analyse the world sheet action of open strings ending on a supertube. Furthermore, as a by-product, we also prove that the supertubes are solutions to all orders in α'. This means that the solution does not have α' corrections but can have string loop corrections. This last point follows from an analysis similar to those of [23, 29, 30].

Our strategy is to consider first a flat supertube with constant magnetic field and then deform it with expectation values for the magnetic field and transverse deformations.

Consider then a supertube extending along directions $X_{1,2}$ and with a field strength:

$$(2\pi\alpha')\, F = dX_0 \wedge dX_1 + B\, dX_1 \wedge dX_2\,, \tag{5.3}$$

(We restore factors of inverse string tension $2\pi\alpha'$ and the string coupling g_s in this section.) The worldsheet action is given by:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu - \frac{i}{2} \int_{\partial\Sigma} F_{\mu\nu} X^\mu \partial_\tau X^\nu \tag{5.4}$$

This is a free theory with boundary propagators given by [31, 32]:

$$\left\langle X^\mu(\tau) X^\nu(\tau') \right\rangle = -\alpha' G^{\mu\nu} \ln\left(\tau - \tau'\right)^2 + \frac{i}{2} \theta^{\mu\nu} \epsilon\left(\tau - \tau'\right), \tag{5.5}$$

where $\epsilon(\tau) = \mathrm{sign}(\tau)$. Also, $G^{\mu\nu}$ and $\theta^{\mu\nu}$ are a symmetric and anti-symmetric matrix respectively and are given by:

$$G^{\mu\nu} + \frac{1}{2\pi\alpha'} \theta^{\mu\nu} = \left(\frac{1}{\eta + 2\pi\alpha' F}\right)^{\mu\nu} \tag{5.6}$$

The propagators for the transverse coordinates are not modified by the gauge field, so we will concentrate on $\mu, \nu = 0, 1, 2$. Using the value of F given by eq. (5.3) we obtain that the non-vanishing components of $G^{\mu\nu}$ and $\theta^{\mu\nu}$ are:[4]

$$\begin{aligned} G^{00} &= -\frac{1+B^2}{B^2}\,, & G^{02} &= G^{20} = -\frac{1}{B}\,, & G^{11} &= \frac{1}{B^2}\,, \\ \theta^{01} &= -\theta^{10} = -\frac{2\pi\alpha'}{B^2}\,, & \theta^{12} &= -\theta^{21} = \frac{2\pi\alpha'}{B}\,. \end{aligned} \tag{5.7}$$

Notice that $G^{22} = 0$ and hence the boundary correlator $\langle X^2(\tau) X^2(\tau') \rangle$ vanishes, as desired. We can also compute the open-string coupling constant G_0 as [22]:

$$G_o = g_s \left(\frac{\det(\eta + 2\pi\alpha' F)}{\det(\eta)}\right)^{\frac{1}{2}} = g_s B\,. \tag{5.8}$$

[4]Related expressions appeared in [8].

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Therefore, we can trust the open-string picture as long as $g_s B \ll 1$ and $g_s \ll 1$ to suppress closed string loops. If we consider N branes then we need $g_s BN \ll 1$.

Going back to the open-string metric, at this point it is convenient to define new coordinates as:

$$\tilde{X}_0 = BX_0 - \frac{1+B^2}{2} X_2 \,, \qquad \tilde{X}_1 = BX_1 \,. \tag{5.9}$$

The open-string metric can be written now as

$$dS^2 = G_{\mu\nu} dX^\mu dX^\nu = -2 d\tilde{X}_0 dX_2 + d\tilde{X}_1^2 \,. \tag{5.10}$$

Notice that $\tilde{X}_0$ and X_2 are both null coordinates in this metric.

Now we can consider deformations of the conformal theory corresponding to the addition of a position-dependent magnetic field, as well as changing the shape of the supertube. The worldsheet action should now include terms

$$S_I = i \int_{\partial\Sigma} A_1(X_2) \partial_\tau X_1 + i \int_{\partial\Sigma} \Phi_i(X_2) \partial_\sigma X_i \,. \tag{5.11}$$

For small values of $A_1(X_2)$ and $\Phi_i(X_2)$ all that is needed is that the extra terms have conformal dimension 1 in the unperturbed theory. This follows from the fact that $\langle X^2(\tau) X^2(\tau')\rangle = 0$ provided that there are no contractions between X_2 and $\partial_\tau X^1$. These contractions are proportional to $\partial_\tau \epsilon(\tau - \tau') = \delta(\tau - \tau')$ and vanish in a point splitting regularization [22].

We want, however, to go a step further and consider the whole perturbative series showing that the β functions β^{A_1}, β^{Φ_i} are 0 to all orders in α'. The basic observation is that X_2 is a null coordinate in the open-string metric (5.10). This means that the background is analogous to a plane wave and we can use the same methods.

One way to show that the fields $A_1(X_2)$ and $\Phi_i(X_2)$ are not renormalized is to expand S_I around a background field $X^\mu = \bar{X}^\mu + x^\mu$. Then one can show that terms proportional to $\partial_\tau \bar{X}^1$ or $\partial_\sigma \bar{X}^i$, which could renormalize $A_1(X_2)$ or $\Phi_i(X_2)$, cannot be generated in one-particle-irreducible vacuum Feynman diagrams since there is no $\langle X^2 X^2\rangle$ propagator.

Another way is to proceed directly to compute the partition function with sources. As discussed in [22] the non-commutativity produces an overall factor in the computation of vertex correlators and so the computations can be done in the commutative case introducing the parameters $\theta^{\mu\nu}$ at the end. Furthermore, we just need to consider the boundary theory since in the bulk of the world sheet there are no divergences. The boundary theory with sources is simply

$$\begin{aligned} S = & \int_{\partial\Sigma,\partial\Sigma'} d\tau\, d\tau'\, X^2_\tau\, G^{-1}_{\tau,\tau'}\, \tilde{X}^0_{\tau'} + \tilde{X}^1_\tau\, G^{-1}_{\tau,\tau'}\, \tilde{X}^1_{\tau'} + \partial_\sigma X^i_\tau \left(\partial_{\tau\tau'} G_{\tau,\tau'}\right)^{-1} \partial_\sigma X^i_{\tau'} + \\ & + \int_{\partial\Sigma} A_1(X_2)\partial_\tau \tilde{X}^1 + i \int_{\partial\Sigma} \Phi_i(X_2) \partial_\sigma X^i + \int_{\partial\Sigma} J_0 G^{-1}_{\tau,\tau'} X^2 + J_2 G^{-1}_{\tau,\tau'} X^0 + \\ & + \int_{\partial\Sigma} J_1 G^{-1}_{\tau,\tau'} X^1 + J_i \left(\partial_{\tau\tau'} G_{\tau,\tau'}\right)^{-1} \partial_\sigma X^i, \end{aligned} \tag{5.12}$$

where $G_{\tau,\tau'} = \ln|\tau - \tau'|$. For the fields X^i the boundary action is in terms of the normal derivative $\partial_\sigma X^i$, since those are the boundary data for a field obeying Dirichlet boundary conditions. As in the case of plane waves [23], the idea is that we can integrate in $\tilde{X}^0$ since it appears linearly in the action. This fixes X^2 to its classical value J_2. Integrating in X^2 amounts to replacing $X_2 \to J_2$. Afterwards all the integrals are gaussian and the interactions are linear in the fields, corresponding to shifting the sources and not introducing any divergences. The β functions will then vanish as we wanted to show. We can also compute string diagrams as in the free theory but we should include the appropriate non-commutative factors. A similar analysis can be done for the superstring with the same result.

6. Discussion

In this paper, we have considered several different aspects of the physics of supertubes. First, we have examined D2 supertubes ending on an orthogonal D4-brane, by an explicit construction of the appropriate field configuration in the worldvolume theory of the D4-brane. From the latter point of view, the supertube appears to be a dyonic string. It would be interesting to consider these objects further as a probe of the gauge theory. For example, in the large N limit, hanging a supertube in the throat geometry of a collection of D4-branes could tell us about the correlation functions of the dyonic strings, using the gravity/gauge theory correspondence [33]. It may also be interesting to study these configurations from the point of view of the D2-brane worldvolume theory. For a collection of coincident D2-branes, this theory becomes non-abelian and the expansion of the supertube into a D4-brane can be realized using non-commutative geometry, following the constructions of [18].

We also considered the higher dimensional generalizations of the supertube. In section 3, a D4 supertube was constructed where the constituent branes included D4-branes, D2-branes and fundamental strings. Using the freedom of rotating the D2-branes, we showed that in this case the Born-Infeld action has supersymmetric solutions describing supertubes with non-flat worldvolume. These solutions are singular because of the infinite extent of the D2-branes, however, we argued that this problem would be avoided if the construction was generalized to a compact space.

In section 4, we constructed a family of D3 configurations where the spatial topology was $S^1 \times S^1 \times R$ and the two orthogonal circles were both supported against collapse by independent angular momenta. These solutions were shown to be stable against small fluctuations but were not supersymmetric. In this respect, these solutions are rather like the nonsupersymmetric configurations of [5] where ellipsoidal membranes were supported by angular momenta in orthogonal planes. While we found a broad family of solutions, it is interesting to note that the profiles of the magnetic field and shape on each of the circles were correlated as in eq. (4.12). Hence one does not seem to have the same arbitrariness as in the case of the D2 supertube.

Finally we examined D2 supertubes from the point of view of the worldsheet theory of open strings. One result was that the supertubes are solutions to all orders in α'. Hence the arbitrariness in choosing the magnetic field and the shape of the supertube is not lifted in string theory (at least at lowest order in g_s) by α' corrections beyond those captured in

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the Born-Infeld action. The way in which this result was realized that the boundary correlators vanished for the coordinate parametrizing the cross-section of the supertube. This was because the boundary correlators are determined by the open-string metric described in [22], which was modified by the background gauge field strengths on the D2-brane. Hence, from this point of view, the arbitrariness in the profile of the supertube is similar to that appearing in the profile for certain exact closed-string backgrounds representing plane gravitational waves [23]. It is also reminiscent of the recent discussion of supertubes given in [9].

We might consider the open-string metric for the configurations of section 2 describing a supertube intersecting an orthogonal D4-brane. To simplify the discussion, consider a flat supertube with constant density ρ, which is a particular case of the examples considered in 2.3. In this case, the gauge and scalar fields are given by:

$$A_0 = -X_9 = -\frac{\pi\rho}{r}, \qquad A_x = \frac{\pi j}{r} \tag{6.1}$$

where we use the notation of subsection 2.3. The induced worldvolume metric and field strength follow as:

$$\begin{aligned} ds^2 &= -dx_0^2 + dx^2 + \left(1 + \frac{\pi^2\rho^2}{r^4}\right) dr^2 + r^2 d\Omega_2^2 \\ F &= -\frac{\pi\rho}{r^2} dx_0 \wedge dr + \frac{\pi j}{r^2} dx \wedge dr + \pi j \sin(\theta) d\theta d\phi \end{aligned} \tag{6.2}$$

The open-string metric is then easily computed as

$$\begin{aligned} dS^2 &= -\frac{1}{f(r)} dx_0^2 + \left(1 + \frac{\pi^2 j^2}{r^4 f(r)}\right) dx^2 - 2\frac{\pi^2 j\rho}{r^4 f(r)} dx_0 dx + \\ &\quad + \left(1 + \frac{\pi^2 j^2}{r^4}\right) dr^2 + r^2 \left(1 + \frac{\pi^2 j^2}{r^4}\right) d\Omega_2^2 \,, \end{aligned} \tag{6.3}$$

where $f(r) = 1 + \pi^2\rho^2/r^4$. It is clear that for $r \to \infty$ this metric is just the flat D4-brane metric, and hence the boundary correlator for the worldsheet x field will take a conventional form. However for $r \to 0$ we obtain

$$dS^2 \simeq \left(1 + \frac{j^2}{\rho^2}\right) dx^2 - 2\frac{j}{\rho} dx_0 dx + \frac{\pi^2 j^2}{r^4} dr^2 + \frac{\pi^2 j^2}{r^2} d\Omega_2^2 \tag{6.4}$$

which, after identifying $B = j/\rho$ and changing coordinates to $X_9 = \pi\rho/r$ agrees both eqs. (5.9), (5.10). In particular, $G^{xx} \simeq 0$ and so the corresponding boundary correlator vanishes. It would be interesting to investigate this issue further by considering how a perturbation propagates from the D4-brane into the supertube spike.

Finally, note that supergravity solutions corresponding to supertubes were constructed [34] and these turned out to be related to the so-called chiral sigma models of [29], as noticed in [35]. In fact the uplifted eleven-dimensional solution of [34] is given by the metric and three form:

$$\begin{aligned} ds_{11}^2 &= U^{-2/3}\left[-dt^2 + dz^2 + K(dt+dz)^2 + 2(dt+dz)A + dx^2\right] + U^{1/3} d\vec{y} d\vec{y} \\ C_{[3]} &= U^{-1} dt \wedge dx \wedge dz - U^{-1}(dt+dz) \wedge dx \wedge A \,, \end{aligned} \tag{6.5}$$

where $\vec{y}$ spans R^8, $U(\vec{y})$ and $K(\vec{y})$ are harmonic functions and $A(\vec{y})$ is a harmonic 1-form in R^8. Dimensionally reducing along z one obtains the supertube solution [34] and dimensionally reducing in x the chiral sigma model (which is exact to all orders in α'). Here, we wish to remark that near the supertube the dilaton diverges and one is forced to use the 11-dimensional perspective. However since $U \to \infty$ in this limit, the size of the x circle becomes small and one can dimensionally reduce in x obtaining a chiral null model as a near horizon description which in fact is just the near horizon limit of the fundamental string solution. One might also observe that in these supergravity solutions, the directions tangent to the cross-section of the supertube become null in the near horizon region. This may be related to the 'null' behavior of the boundary correlators discussed above, as this near horizon geometry should capture the physics of the worldvolume theory.

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